How to Science

Part 2: Our Universe = Math?

Last time, we left off with a mystery: what is the mathematical connection between the lengths and tensions of strings that sound good together?

Let's start with length. The simplest pattern I see in Figure 3 is that strings with the same lengths sound good together – this is not too surprising, since these strings are at the same tension and have the same length, they should make pretty much the same sound. Now, what about our other pairs of strings that sound good together?

If we look at the 3rd row of Figure 3, we see that our second string is fixed to a length of 60 centimeters, and the lengths of our first string that sound good with our second are 60, 45, 40, 30, and 20 centimeters.

Now, what do these numbers have in common?

Well, if we divide the length of our first string by the length of our second string, we see that the strings that sound good together do have something interesting in common – the ratio of their lengths all reduce to simple fractions, while the ratios of the lengths of string that do not sound good together do not reduce to simple fractions![1]

This was the first discovery of Pythagoras – strings sound good together when the ratio of their lengths is simple.

Now, let's have a look at tension. Maybe the same "simple ratio" rule applies? If we divide the tensions of our first string by our second, we see that the tensions that sound good together do result in relatively simple ratios, as shown in figure 6. However, some string tensions that don't sound good together - such as 3.6 and 1.8 kg – also reduce to very simple ratios!

STRING 2 LENGTH (cm) \ STRING 1 LENGTH (cm)	70	65	60	55	50	45	40	35	30	25	20
70	1/1	13/14	6/7	11/14	5/7	9/14	4/7	1/2	3/7	5/14	2/7
65	14/13	1/1	12/13	11/13	10/13	9/13	8/13	7/13	6/13	5/13	4/13
60	7/6	13/12	1/1	11/12	5/6	3/4	2/3	7/12	1/2	5/12	1/3
55	14/11	13/11	12/11	1/1	10/11	9/11	8/11	7/11	6/11	5/11	4/11
50	7/5	13/10	6/5	11/10	1/1	9/10	4/5	7/10	3/5	1/2	2/5
45	14/9	13/9	4/3	11/9	10/9	1/1	8/9	7/9	2/3	5/9	4/9
40	7/4	13/8	3/2	11/8	5/4	9/8	1/1	7/8	3/4	5/8	1/2
35	2/1	13/7	12/7	11/7	10/7	9/7	8/7	1/1	6/7	5/7	4/7
30	7/3	13/6	2/1	11/6	5/3	3/2	4/3	7/6	1/1	5/6	2/3
25	14/5	13/5	12/5	11/5	2/1	9/5	8/5	7/5	6/5	1/1	4/5
20	7/2	13/4	3/1	11/4	5/2	9/4	2/1	7/4	3/2	5/4	1/1

Figure 5 | Simple is Good. If we take Figure 1, divide the length of our first string by the length of our second, and simplify, this is what we get. Just as in figure 1, shaded squares show good sounding string combinations. Notice that good sounding string combinations result in simpler fractions! Thanks to Ruben Peter for creating this tables!

So it seems that the hidden mathematical relationship between string tensions and sound must be a bit deeper.

If we have a closer look at the ratios of the tensions of strings that sound good together, we may see a hint of a deeper pattern. Notice that quite a few of our good sounding tension ratios are exactly equal to perfect squares.

Following this hunch, let's try taking the square root of our tension ratios, the result is shown in Figure 7. Remarkably, this simple transformation snaps our problem into focus – our new square roots ratios are simple when our string combinations sound good together, and complex when our strings don't sound good together.

This was Pythagoras' second remarkable discovery: Strings sound good together when the square roots the ratios of their tensions are simple.

STRING 2 TENSION (kg) \ STRING 1 TENSION (kg)	1.6	1.8	2	2.2	2.4	2.6	2.8	3	3.2	3.4	3.6	3.8	4
1.6	1/1	9/8	5/4	11/8	3/2	13/8	7/4	15/8	2/1	17/8	9/4	19/8	5/2
1.8	8/9	1/1	10/9	11/9	4/3	13/9	14/9	5/3	16/9	17/9	2/1	19/9	20/9
2	4/5	9/10	1/1	11/10	6/5	13/10	7/5	3/2	8/5	17/10	9/5	19/10	2/1
2.2	8/11	9/11	10/11	1/1	12/11	13/11	14/11	15/11	16/11	17/11	18/11	19/11	20/11
2.4	2/3	3/4	5/6	11/12	1/1	13/12	7/6	5/4	4/3	17/12	3/2	19/12	5/3
2.6	8/13	9/13	10/13	11/13	12/13	1/1	14/13	15/13	16/13	17/13	18/13	19/13	20/13
2.8	4/7	9/14	5/7	11/14	6/7	13/14	1/1	15/14	8/7	17/14	9/7	19/14	10/7
3	8/15	3/5	2/3	11/15	4/5	13/15	14/15	1/1	16/15	17/15	6/5	19/15	4/3
3.2	1/2	9/16	5/8	11/16	3/4	13/16	7/8	15/16	1/1	17/16	9/8	19/16	5/4
3.4	8/17	9/17	10/17	11/17	12/17	13/17	14/17	15/17	16/17	1/1	18/17	19/17	20/17
3.6	4/9	1/2	5/9	11/18	2/3	13/18	7/9	5/6	8/9	17/18	1/1	19/18	10/9
3.8	8/19	9/19	10/19	11/19	12/19	13/19	14/19	15/19	16/19	17/19	18/19	1/1	20/19
4	2/5	9/20	1/2	11/20	3/5	13/20	7/10	3/4	4/5	17/20	9/10	19/20	1/1
4.2	8/21	3/7	10/21	11/21	4/7	13/21	2/3	5/7	16/21	17/21	6/7	19/21	20/21
4.4	4/11	9/22	5/11	1/2	6/11	13/22	7/11	15/22	8/11	17/22	9/11	19/22	10/11
4.6	8/23	9/23	10/23	11/23	12/23	13/23	14/23	15/23	16/23	17/23	18/23	19/23	20/23
4.8	1/3	3/8	5/12	11/24	1/2	13/24	7/12	5/8	2/3	17/24	3/4	19/24	5/6
5	8/25	9/25	2/5	11/25	12/25	13/25	14/25	3/5	16/25	17/25	18/25	19/25	4/5
5.2	4/13	9/26	5/13	11/26	6/13	1/2	7/13	15/26	717/914	17/26	9/13	19/26	10/13
5.4	8/27	1/3	10/27	11/27	4/9	13/27	14/27	5/9	16/27	17/27	2/3	19/27	20/27
5.6	2/7	9/28	5/14	11/28	3/7	13/28	1/2	15/28	4/7	17/28	9/14	19/28	5/7
5.8	8/29	9/29	10/29	11/29	12/29	13/29	14/29	15/29	16/29	17/29	18/29	19/29	20/29
6	4/15	3/10	1/3	11/30	2/5	13/30	7/15	1/2	8/15	17/30	3/5	19/30	2/3
6.2	8/31	9/31	10/31	11/31	12/31	13/31	14/31	15/31	16/31	17/31	18/31	19/31	20/31
6.4	1/4	9/32	5/16	11/32	3/8	13/32	7/16	15/32	1/2	17/32	9/16	19/32	5/8

STRING 2 TENSION (kg) \ STRING 1 TENSION (kg)	4.2	4.4	4.6	4.8	5	5.2	5.4	5.6	5.8	6	6.2	6.4
1.6	21/8	11/4	23/8	3/1	25/8	13/4	27/8	7/2	29/8	15/4	31/8	4/1
1.8	7/3	22/9	23/9	8/3	25/9	26/9	3/1	28/9	29/9	10/3	31/9	32/9
2	21/10	11/5	23/10	12/5	5/2	13/5	27/10	14/5	29/10	3/1	31/10	16/5
2.2	21/11	2/1	23/11	24/11	25/11	26/11	27/11	28/11	29/11	30/11	31/11	32/11
2.4	7/4	11/6	23/12	2/1	25/12	13/6	9/4	7/3	29/12	5/2	31/12	8/3
2.6	21/13	22/13	23/13	24/13	25/13	2/1	27/13	28/13	29/13	30/13	31/13	32/13
2.8	3/2	11/7	23/14	12/7	25/14	13/7	27/14	2/1	29/14	15/7	31/14	16/7
3	7/5	22/15	23/15	8/5	5/3	26/15	9/5	28/15	29/15	2/1	31/15	32/15
3.2	21/16	11/8	23/16	3/2	25/16	13/8	27/16	7/4	29/16	15/8	31/16	2/1
3.4	21/17	22/17	23/17	24/17	25/17	26/17	27/17	28/17	29/17	30/17	31/17	32/17
3.6	7/6	11/9	23/18	4/3	25/18	13/9	3/2	14/9	29/18	5/3	31/18	16/9
3.8	21/19	22/19	23/19	24/19	25/19	26/19	27/19	28/19	29/19	30/19	31/19	32/19
4	21/20	11/10	23/20	6/5	5/4	13/10	27/20	7/5	29/20	3/2	31/20	8/5
4.2	1/1	22/21	23/21	8/7	25/21	26/21	9/7	4/3	29/21	10/7	31/21	32/21
4.4	21/22	1/1	23/22	12/11	25/22	13/11	27/22	14/11	29/22	15/11	31/22	16/11
4.6	21/23	22/23	1/1	24/23	25/23	26/23	27/23	28/23	29/23	30/23	31/23	32/23
4.8	7/8	11/12	23/24	1/1	25/24	13/12	9/8	7/6	29/24	5/4	31/24	4/3
5	21/25	22/25	23/25	24/25	1/1	26/25	27/25	28/25	29/25	6/5	31/25	32/25
5.2	21/26	11/13	23/26	12/13	25/26	1/1	27/26	14/13	29/26	15/13	31/26	16/13
5.4	7/9	22/27	23/27	8/9	25/27	26/27	1/1	28/27	29/27	10/9	31/27	32/27
5.6	3/4	11/14	23/28	6/7	25/28	13/14	27/28	1/1	29/28	15/14	31/28	8/7
5.8	21/29	22/29	23/29	24/29	25/29	26/29	27/29	28/29	1/1	30/29	31/29	32/29
6	7/10	11/15	23/30	4/5	5/6	13/15	9/10	14/15	29/30	1/1	31/30	16/15
6.2	21/31	22/31	23/31	24/31	25/31	26/31	27/31	28/31	29/31	30/31	1/1	32/31
6.4	21/32	11/16	23/32	3/4	25/32	13/16	27/32	7/8	29/32	15/16	31/32	1/1

Figure 6 | Not Quite. Unlike in Figure 5, tension combinations that sound good together do not reduce to simple fractions when divided. This is not the hidden mathematical connection between good sounding strings combinations and string tension. However, there is a hint here!

1 "Simple" is, of course, a bit relative here! For now, let's call any simplified fraction "simple", if its denominator is less than 6.

	STRING 1 TENSION (kg)																								
STRING 2 TENSION (kg)	1.6	1.8	2	2.2	2.4	2.6	2.8	3	3.2	3.4	3.6	3.8	4	4.2	4.4	4.6	4.8	5	5.2	5.4	5.6	5.8	6	6.2	6.4
1.6	1/1	612/577	341/305	428/365	1079/881	914/717	1012/765	571/417	1393/985	1121/769	3/2	1481/961	1140/721	1220/753	1257/758	1075/634	1351/780	1393/788	1170/649	1376/749	1738/929	1883/989	1921/992	250/127	2/1
1.8	544/577	1/1	760/721	419/379	1142/989	780/649	1009/809	559/433	4/3	1362/991	1393/985	1004/691	963/646	333/218	1207/772	1151/720	1277/782	5/3	1562/919	1351/780	1307/741	1743/971	922/505	1520/819	1665/883
2	305/341	684/721	1/1	924/881	505/461	911/799	1001/846	1079/881	912/721	781/599	1292/963	1366/991	1393/985	755/521	531/358	549/362	1055/681	1140/721	1269/787	1515/922	1593/952	1565/919	1351/780	206/117	1669/933
2.2	365/428	379/419	840/881	1/1	1010/967	599/551	537/476	1079/924	439/364	409/329	1095/856	1309/996	716/531	1057/765	1393/985	1138/787	387/262	1197/794	226/147	1562/997	1262/791	1083/667	1085/657	695/414	365/214
2.4	396/485	808/933	461/505	922/963	1/1	994/955	701/649	341/305	1142/989	951/799	1079/881	1174/933	559/433	1012/765	524/387	1192/861	1393/985	1351/936	1391/945	3/2	333/218	869/559	1140/721	1281/797	1277/782
2.6	717/914	540/649	799/911	551/599	955/994	1/1	1017/980	811/755	720/649	948/829	919/781	1163/962	996/803	924/727	147/113	967/727	1159/853	900/649	1393/985	1408/977	1155/787	1395/934	477/314	227/147	1525/972
2.8	542/717	360/449	715/846	476/537	649/701	927/962	1/1	1032/997	929/869	962/873	271/239	798/685	551/461	1079/881	791/631	737/575	999/763	1049/785	1003/736	961/692	1393/985	1291/897	1316/899	747/502	765/506
3	417/571	433/559	396/485	769/898	305/341	755/811	940/973	1/1	63/61	1055/991	505/461	305/271	1142/989	1001/846	1073/886	977/789	912/721	559/433	628/477	1292/963	1231/901	1193/858	1393/985	1002/697	1427/977
3.2	408/577	3/4	570/721	364/439	808/933	585/649	869/929	945/976	1/1	67/65	612/577	753/691	341/305	881/769	428/365	1151/960	1079/881	5/4	914/717	947/729	1012/765	381/283	571/417	380/273	1393/985
3.4	627/914	593/815	599/781	329/409	799/951	829/948	873/962	480/511	65/67	1/1	71/69	684/647	961/886	379/341	463/407	221/190	985/829	593/489	1092/883	799/634	471/367	943/722	631/475	1241/919	911/664
3.6	2/3	408/577	682/915	351/449	396/485	781/919	239/271	461/505	544/577	69/71	1/1	75/73	760/721	701/649	419/379	1049/928	1142/989	713/605	780/649	1079/881	1009/809	740/583	559/433	500/381	4/3
3.8	523/806	404/587	695/958	751/987	662/833	761/920	588/685	271/305	691/753	612/647	73/75	1/1	79/77	840/799	382/355	461/419	1025/912	1123/979	1123/960	602/505	261/215	488/395	1053/838	479/375	815/628
4	456/721	646/963	408/577	531/716	433/559	803/996	461/551	808/933	305/341	886/961	684/721	77/79	1/1	83/81	924/881	563/525	505/461	341/305	911/799	1055/908	1001/846	637/529	1079/881	559/449	912/721
4.2	545/883	218/333	521/755	448/619	542/717	572/727	396/485	715/846	769/881	341/379	649/701	760/799	81/83	1/1	87/85	1012/967	929/869	551/505	780/701	271/239	1142/989	208/177	551/461	989/814	753/610
4.4	439/728	449/702	358/531	408/577	387/524	113/147	631/791	687/832	365/428	407/463	379/419	355/382	840/881	85/87	1/1	91/89	1010/967	856/803	599/551	524/473	537/476	1124/979	1079/924	165/139	439/364
4.6	588/997	431/689	362/549	500/723	437/605	727/967	575/737	789/977	769/922	190/221	882/997	419/461	525/563	924/967	89/91	1/1	95/93	1004/963	185/174	649/599	822/745	996/887	1053/922	339/292	611/518
4.8	571/989	297/485	559/866	262/387	408/577	315/428	333/436	570/721	396/485	829/985	808/933	799/898	461/505	869/929	922/963	93/95	1/1	99/97	994/955	612/577	701/649	1019/927	341/305	1024/901	1142/989
5	211/373	3/5	456/721	463/698	415/599	468/649	336/449	433/559	4/5	489/593	605/713	843/967	305/341	505/551	803/856	916/955	97/99	1/1	103/101	1033/994	835/789	755/701	505/461	304/273	999/883
5.2	360/649	303/515	498/803	147/226	214/315	408/577	419/571	477/628	717/914	714/883	540/649	854/999	799/911	630/701	551/599	174/185	955/994	101/103	1/1	107/105	1017/980	941/891	811/755	487/446	720/649
5.4	264/485	571/989	227/373	637/998	2/3	671/967	692/961	682/915	729/947	634/799	396/485	505/602	803/933	239/271	473/524	599/649	544/577	943/980	105/107	1/1	111/109	997/962	760/721	869/811	749/688
5.6	240/449	271/478	551/922	571/911	218/333	571/838	408/577	658/899	542/717	367/471	360/449	215/261	715/846	808/933	476/537	745/822	649/701	789/835	927/962	109/111	1/1	115/113	1032/997	262/249	929/869
5.8	406/773	283/508	340/579	380/617	559/869	545/814	107/154	607/844	283/381	722/943	583/740	395/488	529/637	177/208	844/969	887/996	796/875	701/755	891/941	962/997	113/115	1/1	119/117	1006/973	583/555
6	63/122	505/922	571/989	416/687	456/721	314/477	166/243	408/577	417/571	475/631	433/559	343/431	396/485	461/551	769/898	739/844	305/341	461/505	755/811	684/721	940/973	117/119	1/1	123/121	63/61
6.2	127/250	465/863	117/206	414/695	275/442	147/227	502/747	681/979	273/380	391/528	381/500	375/479	449/559	814/989	139/165	292/339	608/691	273/304	446/487	811/869	249/262	914/945	121/123	1/1	127/125
6.4	1/2	306/577	341/610	214/365	297/485	457/717	506/765	571/834	408/577	664/911	3/4	628/815	570/721	610/753	364/439	518/611	808/933	883/999	585/649	688/749	869/929	555/583	945/976	125/127	1/1

Figure 7 | Heck Yeah! The square roots of the tension ratios shown in Figure 6 reveal the hidden mathematical connection we were looking for! Strings sound good together when the square root of the ratio of their tensions is simple. Note that many of the fractions here are approximate representation of square roots.

This information was helpful for early instrument makers, but what's really interesting here is how these discoveries changed the way we humans think about the universe we live in.

These discoveries, along with another interesting Pythagorean discovery involving right triangles, really got Pythagoras and his followers thinking.

Why is it that mathematics is able to predict what we observe in the world around us?

Division, square roots, fractions, and even numbers kind of seem like human inventions – why are they showing up so clearly in triangles and vibrating strings?

These mysteries were compelling enough to convince the Pythagoreans of something that might sound a little far fetched today - that our world is literally built from mathematics.

Figure 8 | No, way right? There's no way our universe and the world we live in are actually made from...math, right?

This may sound ridiculous, but we don't actually have to look very far to find very bright modern day physicists who believe this.[2]

Now, whether or not our universe is built from math is still open for debate, but what I think is really interesting here is what we do with this mystery.

2 "I argue that with a sufficiently broad definition of mathematics, it implies the Mathematical Universe Hypothesis (MUH) that our physical world is an abstract mathematical structure." - Physicist Max Tegmark, in The Mathematical Universe

For the Pythagoreans, this was basically it. Why does mathematics show up in vibrating strings and triangles? Because the universe is built from numbers. Mystery solved.[3]

It would take a couple thousand years for us humans to really start probing what I think is the more interesting question – how deep does the connection between mathematics and our universe go?

If we can predict when two strings will sound good together, what else can we predict?

Figure 9 | How Deep? Even if our universe isn't made from math, there's clearly a connection here. How deep does it go?

In the case of our vibrating strings - what actually makes our strings sound good or bad together?

Can we use math to understand what really happens when a string vibrates?

Answering questions like this required one more piece of the scientific puzzle that the Pythagoreans never really committed to – experimentation. It may seem obvious to us now, but it took quite some time for us humans to really accept what is perhaps the most important idea in all of science.

3 The full story here, as usual, is a bit more complex, and quite a bit of the history surrounding Pythagoras and the Pythagoreans is unclear. For example, Pythagoras my have taught that all things are in accordance with number, rather than being literally made from numbers. The Wikipedia articles on Pythagoras and Pythagoreanism are great places to learn more.

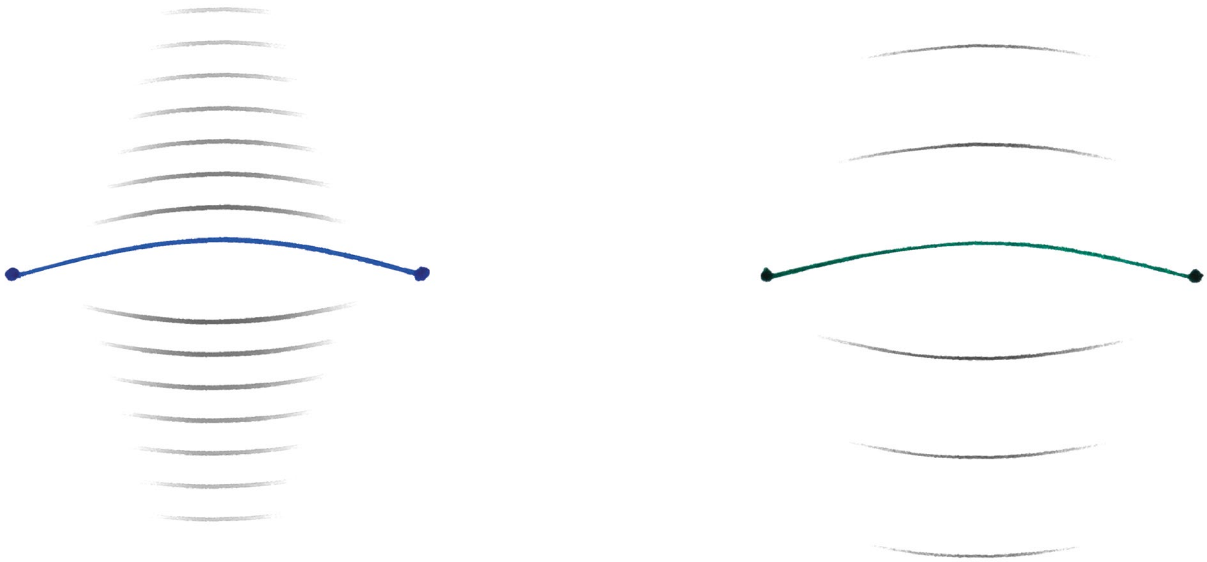

Figure 10 | How Sound Works? Giambattista Benedetti suggested that how high or low a note sounds to us is a result of how frequently a series of rapid pulse arrive.

If it disagrees with experiment…then it's wrong.

When this idea finally started to catch on in 16th century Europe, scientists were finally able to dig deeper into all kinds of mysteries, including the vibrating string.

The first real progress came in the form of an educated guess from the Italian scientist Giambattista Benedetti. Benedetti suggested that musical sounds travel through the air as a series of rapid pulses, and that how high or low a note sounds to us - its pitch – is a direct result of how frequently these pulses arrive. So if Benedetti's suggestion is true, then the sound we hear form a vibrating string is a direct result of how frequently the string moves back and forth also know as its frequency of vibration.

This idea raised all kinds of new questions:

How is a strings' frequency of vibration connected to its length and tension?

Does anything other length and tension effect a strings' frequency?

How can we measure the frequency of real strings, when they vibrate back and forth way too quickly for us to see?

In the late 1500s, around 50 years after Benedetti guessed that pitch was a direct result of frequency, the great Italian scientist Galileo Galilei turned his attention to these questions. Guided by the work of his father Vincenzio, Galileo made some well-informed guesses at our first two questions that would ultimately turn out to be correct.

However, Galileo also claimed that since strings vibrate too fast to see, actually measuring their frequency was impossible[4] – meaning that he had no experimental means

GALILEO GALILEI

1564-1642

Figure 11 | Galileo Galileo. One of the earliest scientist to achieve Rihanna-level fame by going by only one name: Galileo.

to test his guesses.

But fortunately for us, Galileo was wrong.

Within thirty years of Galileo's work, the French Priest and Scientist Marin Mersenne did measure the frequency of vibrating strings, and was able to experimentally confirm Galileo's guesses. Mersenne was even able to use his methods to compute the frequency of pipe organ notes with around 90% accuracy.[5]

So, how did Mersenne do what Galileo said was impossible?

What do you think?

Given the technology available in the early 1600s, how would you try to figure out how the rate of vibration of a string - its frequency - depends on its length or tension?

How would you prove Galileo wrong? [6]

4 Cohen, H.F. (2013). Quantifying Music: The Science of Music at the First Stage of Scientific Revolution 1580–1650, p.101.

5 Computed from Origins in Acoustics by Frederick Vinton Hunt, page 93.

6 See next page for a small hint!

Small Hint

In the 1600s it was possible to measure the passage of time with reasonable accuracy using a pendulum. Thanks to the work of Galileo himself, it was known that for small movements, a pendulum would swing back and forth at a reasonably constant rate, that depended only on its length. Sixteenth century scientists often timed experiments by counting pendulum swings.

Recommended Reading

1. Frank Wilcenz, *A Beautiful Question*. Penguin Books.
2. Max Tegmark, Our Mathematical Universe. Vintage Books.