How to Science

Part 4: Science

Last time, we left off trying to do what Galileo did – guess the connection between the length and frequency of a vibrating string.

From our observations in Table 1, it's pretty clear that as the length of our string goes up, its frequency goes down, and as the length of our string goes down, it's frequency goes up. However, if we're going to make specific predictions about the frequency of a given string, we need a more specific guess.

Notice that when the length of our string doubles from 40 to 80 cm, our frequency is roughly cut in half. So when our length is multiplied by a factor of two, our frequency appears to be divided by a factor of two. And further, when the length of our string is divided by a factor of two from 40 to 20 cm, our frequency is approximately multiplied by a factor of two.

How can we capture this relationship mathematically? What mathematical relationship turns two into one half and one half into two?

The operation we need here, is the inverse.[1] One divided by our length factor gives us our frequency factor. Mathematically, we can say that frequency is inversely proportional to length.

$$f \propto \frac{1}{L} \tag{1}$$

This was Galileo's guess.

Let's see how Galileo's guess holds up to Mersenne's experiment. As Feynman told us, we need to compute the predictions made by our guess before we compare to experiment. Last time we setup a second string with a tension of 3200 g, and observed a frequency of about 174 Hz at a length of 40 cm, as shown in Table 2. Now, using our guess, let's predict the frequency of this string at a length of 80, 50, and 20 cm.

Since going from our original length of 40 to 80 cm requires us to double our length, according to our educated guess, our frequency should be cut in half. So our original frequency of 174 Hz should become 87 Hz. This is our first prediction.

The next prediction we need to make is for a string

1 Reciprocal, really.

STRING 1 tension = 1600g		STRING 2 tension = 3200g	
LENGTH (cm)	FREQUENCY (Hz)	LENGTH (cm)	FREQUENCY (Hz)
80	65	80	?
40	122	50	?
20	245	40	174
		20	?

Table 1 | Can You See What Galileo Did? What is the connection between the length of a string, and it's frequency of vibration?

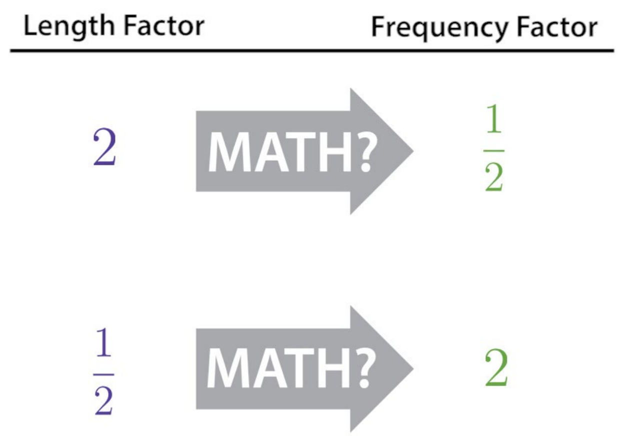

Figure 16 | Can you name the missing mathematics? What mathematical operation turns 2 into 1/2 and 1/2 into 2?

length of 50 cm. Since we're starting at 40 cm, changing our length to 50 cm is not a clean doubling or halving as before – so we might have to actually do some math!

Statements of proportionality like ours can be a little inconvenient to work with, so let's change how we write our expression. Saying that one variable is proportional to another really just means that there's some constant out there, that when multiplied by one variable, always gives us

y is proportional to x	$y \propto x$	$y = kx$

Figure 17 | Different Names for the Same Thing. All of these statements are equivalent, but the one on the right is a smidge easier to work with.

the other. If we call this constant of proportionality k, we can rewrite our statement of proportionality as an equation:

$$f = \frac{k}{L} \tag{2}$$

This equation says the same exact thing as our original proportionality statement in equation (1), except now we've expressed our constant of proportionality, k, explicitly – making our relationship easier to work with. Mathematically, this formula completely captures the concept of inverse proportionality – so if someone tells you that two variables are inversely proportional, assuming they know what their talking about, they don't just mean that when one gets bigger, the other gets smaller, the mean exactly this. That one variable equals some constant of proportionality divided by the other.

Now that we have a slightly more useful version of our mathematical relationship – let's use it! We can use our initial string configuration - length equals 40 cm and frequency equals 174 Hz - to compute our proportionality constant k.

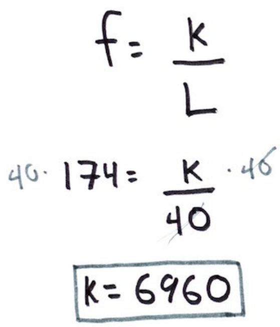

Figure 19 | Computing k. We can compute k for our second string configuration using our frequency and length observations in Table 1. Note that each time we change strings or tension - we need to recompute k!

Now that we have k, we can use it, along with our formula to make predictions for other string lengths at this tension.[2] Plugging in our last two experimental lengths, 50 and 20 cm, we compute 139.2 and 348 Hz for our predicted frequencies, as shown in Figure 20.

Alright, after all that work we now have a guess at our hidden mathematical connection – frequency is inversely proportional to length – and some specific predictions, 87, 139.2, and 348 Hz, computed using our guess.

It's finally time to experiment.

Using our slow motion camera and counting vibrations, we measure frequencies of 91 Hz for our 80 cm string, 141 Hz for our 50 cm string, and 343 Hz for our 20 cm string. And if we compare our results to our predictions, it looks like our predictions are pretty good! In fact, our percent errors are 4.6, 1.3, and 1.4 percent – not bad at all - and thanks to our high speed camera, our results are actually

2 If we change the tension of our string, we have to re-compute k!

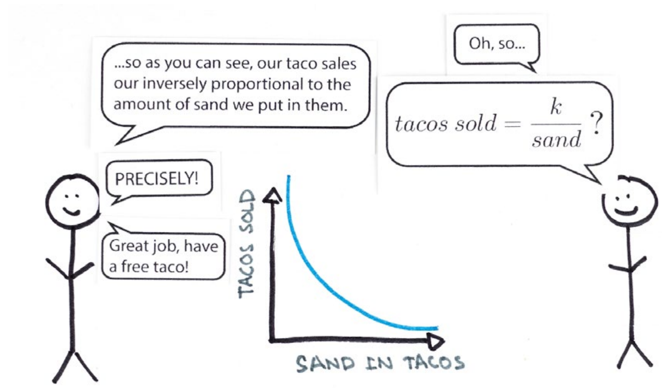

Figure 18 | Inverse Proportionality Means Exactly Equation 2. Mmmmmm.... sand tacos.

LENGTH (cm)	OBSERVED FREQUENCY (Hz)	PREDICTED FREQUENCY (Hz)	% ERROR
80	91	87	4.6
50	141	$\frac{6960}{50} = 139.2$	1.3
40	174		
20	343	$\frac{6960}{20} = 348$	1.4

Figure 20 | Pretty Good! Our predicted frequencies are quite close to our observed frequencies.

more accurate than Mersenne's really long string and pendulum method.[3]

We have found our first scientific law. The frequency of a vibrating string is inversely proportional to its length. This relationship is known today as Mersenne's first law.

And more importantly, we've found a method, the scientific method, that allows us to make these types of discoveries. That through observation, guessing, and experimentation, allows us to explore the connection between mathematics and the physical world. That allows us to really probe the universe we live in.

Now, Galileo's guesses and Mersenne's experiments are really just the beginning of the mysteries of the vibrating string. In the following 18th century, the brightest mathematicians and scientists of the day battled over a more

3 Frederick Vinton Hunt, Origins in Acoustics, p 90-100

complete mathematical description of the vibrating string. Resolving this controversy would require the creation of some very slick and incredibly useful mathematics - differential equations and Fourier analysis.

We'll save these stories for another day, and for now, we have two final mysteries to solve.

We've discovered Mersenne's first law, and thanks to Mersenne's experimental setup, he was able to discover two more laws governing the behavior of the vibrating string. Two more connections between mathematics and our universe.

Mersenne's second law tells us about the connection between a vibrating string's tension and frequency, and Mersenne's third law is about the connection between a string's mass and frequency.

Let's see if we can discover Mersenne's final two laws using the scientific method. Just as before, we'll start by making some observations. First, we'll fix the length of our string to 60 cm, and measure the frequency as we change tensions. This relationship is a little trickier than our length vs frequency relationship, so we'll take more observations this time, as shown in Table 2. Just as before, we'll go ahead and setup a second string configuration that we'll use to test our guesses next time. We'll make the length of this string 40 cm, and take one observation at a tension of 1000g. We'll use our educated guess to make frequency predictions for this string at tensions of 4000, 5000, and 9000 g.

Mersenne's Laws

(1) $$f = \frac{k}{L}$$ Mathematical connection between length and frequency

(2) Mathematical connection between tension and frequency

(3) Mathematical connection between mass and frequency

Figure 21 | One of Mersenne's Three Laws. Thanks to Mersenne's clever experimental setup, we was able to find three very specific connections between mathematics and our universe. Interestingly, all three of these laws were correctly guessed by Galileo *before* Mersenne's experiments. This really makes me wonder just how Galileo had the intuition to figure these relationships out without clear experimental evidence!

Next, we'll explore one last connection between mathematics and vibrating strings – the connection between a string's mass and frequency of vibration. The guitar strings we're using here come in 6 varieties. We'll measure the mass of each string – but we need to be a little careful here – the mass of an entire string depends on its whole length – but in our experiments we really only care

STRING 1
length = 60 cm

TENSION (g)	FREQUENCY (Hz)	TENSION (g)	FREQUENCY (Hz)
1000	66.6	5500	152.2
1500	81.3	6000	162.6
2000	92.2	6500	168.2
2500	102.0	7000	174.4
3000	114.2	7500	177.6
3500	123.0	8000	184.4
4000	133.2	8500	191.9
4500	141.0	9000	195.7
5000	145.3	9500	204.1

STRING 2
length = 40 cm

TENSION (g)	FREQUENCY (Hz)
1000	99.9
4000	?
5000	?
9000	?

Table 2 | Tension vs Frequency. Can you figure out the hidden mathematical connection between tension and frequency? Using your answer, what are your predictions for string 2 under tensions of 4000, 5000, and 9000g?

about the section of the string we allow to freely vibrate. And we don't really want to have to re-compute the mass of the vibrating section of the string each time we change its length.

We can fix this problem by dividing each mass measurement by the full length of each string. The resulting numbers, our mass per unit length[4], tells us the mass of each 1 cm section of our string – allowing us to ignore the effect of the overall length of our strings – and compare strings of different sizes more directly. The mass per unit length of each of our 6 strings is shown in Table 3.[5]

Now, for a given length and tension, we'll measure the frequency of each string and record our results in table 4. And just as before we'll setup a second configuration that we'll use to test our guesses. We now have some nice observations to help us make educated guesses at Mersenne's final two laws.

Now, what do you think?

What is the hidden mathematical connection between a vibrating string's tension and frequency?

What is the hidden mathematical connection between a vibrating string's mass per unit length and frequency?

STRING	MASS (g)	LENGTH (cm)	MASS/LENGTH (g/cm)
E	6.164	101.0	6.10×10^{-2}
A	4.041	102.0	3.96×10^{-2}
D	2.340	100.5	2.33×10^{-2}
G	0.939	106.0	8.86×10^{-3}
B	0.627	105.0	5.97×10^{-3}
E	0.477	105.0	4.54×10^{-3}

Table 3 | Mass Per Unit Length is the Way to Go. We would like to understand how the mass of our vibrating string affects its frequency. However, every time we change the length of our string, the mass of the vibrating portion changes! We can fix this problem by using mass/length instead of the mass of the total length of our string. This is equivalent to linear density.

SETUP 1 (length = 60 cm, tension = 1600 g)		SETUP X (length = 60 cm, tension = 3200 g)		SETUP 2 (length = 40 cm, tension = 1600 g)	
MASS/LENGTH (g/cm)	FREQUENCY (Hz)	MASS/LENGTH (g/cm)	FREQUENCY (Hz)	MASS/LENGTH (g/cm)	FREQUENCY (Hz)
6.10×10^{-2}	84.4	6.10×10^{-2}	117.5	6.10×10^{-2}	123.6
3.96×10^{-2}	105.2	3.96×10^{-2}	149.9	3.96×10^{-2}	?
2.33×10^{-2}	137.0	2.33×10^{-2}	196.6	2.33×10^{-2}	?
8.86×10^{-3}	222.0	8.86×10^{-3}	311.4	8.86×10^{-3}	?
5.97×10^{-3}	272.5	5.97×10^{-3}		5.97×10^{-3}	?
4.54×10^{-3}	324.0	4.54×10^{-3}		4.54×10^{-3}	?

Table 5 | What is the Mathematical Connection Between Mass and Frequency? What do you think the mathematical connection is between a vibrating strings mass per unit length and frequency? Using your answer, what are your predictions for the missing frequency values in setup 2? We've included a little extra data here from a 3rd setup to help.

4 Also known as linear density

5 In an effort to keep these numbers reasonably accurate, the masses shown in Table 4 have been corrected for the small plastic or brass pieces at the end of each string. The brass rings are difficult to remove from the wound strings, however the plastic end pieces come off quite easily. I measured the mass of the plastic end piece of the G string to be 0.036g, and subtracted this value from all string mass measurements. I'm sure that not all of the end peices weight 0.036 grams, but this seems like a reasonable appoximation, and made a moderate improvement to the consistency of the observations.