How to Science

Part 5: Mathematics

Last time, we left off trying to figure out Mersenne's final two laws.

Let's start with the connection between the tension and frequency of a vibrating string. Unlike our length and frequency relationship, tension and frequency seem to vary directly – when tension increases, so does frequency. A reasonable guess then might be that tension and frequency are directly proportional. This would mean that if tension doubled, frequency would also double.

However, our data helps us quickly reject this hypothesis – when our tension doubles from 1000 to 2000 grams, our frequency only increases by a factor of about 1.4, it doesn't double.

Now, when *does* our frequency double? A look at Table 2 shows that our frequency approximately doubles from 66.6 to 133.2 Hz. Notice that making our frequency double required us to increase our tension by a factor of 4.

Let's continue our search by looking for examples of our frequency approximately tripling. Our frequency does roughly triple from 66.6 to 195.7 Hz, and notice that making this jump requires our tension to be increased by a factor of 9, from 1000 to 9000g.

Now we're really getting somewhere – what mathematical relationship between tension and frequency would turn factors of 4 into factors of 2 and factors of 9 into factors of 3?

The piece of mathematics we're looking for here is the square root! [1]

1 This is very similar to the square root relationship we found back in part 2 between the tensions of strings that sound good together...

So, based on this evidence, our hypothesis, our educated guess, is that the frequency of a vibrating string is proportional to the square root of its tension:

$$f \propto \sqrt{T} \qquad (3)$$

We can use our guess to make specific predictions about our untested string setup. Our 40 mm long string, under a tension of 1000 g, vibrates at a frequency of 99.9 hz, as shown in Table 2. Using our educated guess that the frequency of a vibrating string is proportional to the square root of its tension – we can predict that if we quadruple the string tension to 4000 g, our frequency should increase by a factor of the square root of 4, so a factor of 2, making our predicted frequency 199.8 Hz.

Now, what about the other tensions we would like to make predictions for? Just as in our length and frequency relationship, we can make our calculations simpler by figuring out a formula instead of just a proportion. In this case, since we're claiming that frequency is proportional to the square root of tension, we can write that frequency equals some constant of proportionality, k, times the square root of tension:

$$f = k\sqrt{T} \qquad (4)$$

Equations (3) and (4) say the same exact thing, but

STRING 1
length = 60 cm

TENSION (g)	FREQUENCY (Hz)	TENSION (g)	FREQUENCY (Hz)
1000	66.6	5500	152.2
1500	81.3	6000	162.6
2000	92.2	6500	168.2
2500	102.0	7000	174.4
3000	114.2	7500	177.6
3500	123.0	8000	184.4
4000	133.2	8500	191.9
4500	141.0	9000	195.7
5000	145.3	9500	204.1

STRING 2
length = 40 cm

TENSION (g)	FREQUENCY (Hz)
1000	99.9
4000	?
5000	?
9000	?

Table 2 | Tension vs Frequency. Can you figure out the hidden mathematical connection between tension and frequency? Using your answer, what are your predictions for string 2 under tensions of 4000, 5000, and 9000g?

TENSION (g)	OBSERVED FREQUENCY (Hz)	PREDICTED FREQUENCY (Hz)	% ERROR
1000	99.9	99.9	–
4000	191.8	199.8	4.0
5000	218.0	3.159·√5000 = 223.4	2.4
9000	299.6	3.159·√9000 = 299.7	0.03

Figure 22 | Our Equation Works! After computing our constant of proportionality k using our single observation at this length and tension - we've made predictions for all string lengths. And when we compare to experiment using our high speed camera - our predictions are quite close to our observations! Booya!

representing our constant of proportionality explicitly, as k, makes our proportion easier to work with. We can use our observation at this length to compute k, and then simply plug in to our formula to make the rest of our predictions, as shown in Figure 22.

Now that we have a set of specific predictions that we computed using our hypothesis that frequency is proportional to the square root of tension, we can test our hypothesis.

In the 40 cm case, for a tension of 4000 grams, our hypothesis predicts a frequency of 199.8 Hz. And if we set our string to this tension, and carefully measure our frequency, we observe approximately 191.8 vibrations a second, pretty close to our predicted frequency. And as we continue testing our predictions, we see good agreement between our observations and predictions!

We have found Mersenne's second law. One more hidden connection between mathematics and the physical world. The frequency of a vibrating string is directly proportional to the square root of its tension.

Now, thanks to Mersenne's clever experimental setup, he was able to confirm one more guess from Galileo's about the connection between the mass and frequency of our vibrating string.

If we have a quick look at our mass and frequency data from last time (Table 5 below), we see that as our mass per unit length decreases, our frequency increases. This is pretty reasonable – lighter string strings make higher pitched sounds.

Now, how exactly does frequency increase as mass decreases? As we figured out last time, frequency also increases as length decreases, and these two quantities are inversely proportional. Frequency is equal to some constant k divided by length. Let's see if this rule fits our frequency vs mass data:

$$f = \frac{k}{M}? \tag{5}$$

This data is a little harder to work with than our last two

SETUP 1 (length = 60 cm, tension = 1600 g)		SETUP X (length = 60 cm, tension = 3200 g)		SETUP 2 (length = 40 cm, tension = 1600 g)	
MASS/LENGTH (g/cm)	FREQUENCY (Hz)	MASS/LENGTH (g/cm)	FREQUENCY (Hz)	MASS/LENGTH (g/cm)	FREQUENCY (Hz)
6.10×10^{-2}	84.4	6.10×10^{-2}	117.5	6.10×10^{-2}	123.6
3.96×10^{-2}	105.2	3.96×10^{-2}	149.9	3.96×10^{-2}	?
2.33×10^{-2}	137.0	2.33×10^{-2}	196.6	2.33×10^{-2}	?
8.86×10^{-3}	222.0	8.86×10^{-3}	311.4	8.86×10^{-3}	?
5.97×10^{-3}	272.5	5.97×10^{-3}		5.97×10^{-3}	?
4.54×10^{-3}	324.0	4.54×10^{-3}		4.54×10^{-3}	?

Table 5 | What's up with Mass and Frequency? What do you think the mathematical connection is between a vibrating strings mass per unit length and frequency? Using your answer, what are your predictions for the missing frequency values in setup 2? We've included a little extra data here from a 3rd setup to help.

datasets, because we don't have as much control over our experiment. We were able to choose almost any value we wanted for our string length and tension, while the mass per unit length of our strings is limited to the 6 different guitar strings in our experiment. This makes it highly unlikely that we will simply be able to look for examples in our data where our quantities are double or tripled as we did before.[2]

This means that identifying patterns in our data by just looking at it is probably not going to work.[3] And fortunately, we don't have to rely on staring down our data and hoping for magical insights.

Instead, we can use a little math to help us search for patterns. First we would like to know if the same rule of inverse proportionality we used for length and frequency,

$$f = \frac{k}{L} \tag{2}$$

works for our mass and frequency data. In our previous two examples, we used our rules to make specific predictions for experimental setups we hadn't yet observed – and if our predictions were reasonably close to our observations, that meant that our rule was probably correct. We can apply the same exact idea here, except we'll use the rule we would like to test to make predictions about the data that we already have. If our predictions match our data, than we should have a good rule!

To test our potential rule, $f=k/M$, we'll compute k using one observation – and using this value of k, we'll make predictions for the rest of our observations:

$$f = k/M$$

$$84.4 = k/6.10\times10^{-2}$$

$$k = 84.4 \cdot 6.10\times10^{-2}$$

$$\boxed{k = 5.148}$$

So, how well do our predictions match our observations?

Ehhh, not so well – check out Figure 24. Up until this point our observations and prediction have been within about 5% of each other, and here our predictions are up to 70% off.

We've guessed the wrong rule. Frequency in not inversely

$$\cancel{f = \frac{k}{M}?} \tag{5}$$

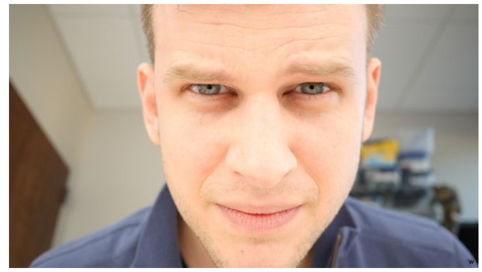

Figure 23 | Tried Staring Down our Mass vs. Frequency Data. Didn't work.

proportional to string mass.

So, what should we do next? Mathematics is a pretty big topic, there's literally thousands of rules we could guess next.

Well, instead of guessing randomly, let's try to be a little more systematic. While our first guess was up to 70% off, it did capture the inverse relationship between mass and frequency – as mass decreases, both our observed and predicted frequency increase. The part of our equation responsible for this behavior is the division, so we should probably hang on to this.

SETUP 1
length = 60 cm
tension = 1600g

MASS/LENGTH (g/cm)	OBSERVED FREQUENCY (Hz)	f = K/M PREDICTION	% ERROR
6.10×10^{-2}	84.4	84.4	–
3.96×10^{-2}	105.2	$5.148/3.96\times10^{-2}$ = 130.0	19.1
2.33×10^{-2}	137.0	$5.148/2.33\times10^{-2}$ = 221.0	38.0
8.86×10^{-3}	222.0	$5.148/8.86\times10^{-3}$ = 581.1	61.8
5.97×10^{-3}	272.5	$5.148/5.97\times10^{-3}$ = 862.4	68.4
4.54×10^{-3}	324.0	$5.148/4.54\times10^{-3}$ = 1134.0	71.4

Figure 24 | Our First Rule Sucks! Our first guess about the mathematical connection between frequency and mass, $f=k/M$, is super wrong. We should probably give up.

Now, notice that our predicted frequencies increase way more quickly than our observed frequencies. How can we modify our guess to make frequency increase more slowly as mass decreases?

Well, there are an infinite number of ways we could modify our guess, but let's start with something simple. One way to make a function grow more slowly is to take its square root:[4]

$$f = \sqrt{\frac{k}{M}}? \tag{6}$$

2 Of course, when Galileo correctly guessed this mathematical relationship, he had even less information!
3 not for me at least! :)

4 sortof..the complication here is that for function values < 1, taking the square root actually makes our function larger. Fortunately, since our frequency predictions are significantly larger than 1, the square root has the desired effect.

Further, square roots have already shown up twice in our mathematical exploration of the vibrating string, so this may be a reasonable guess.[5]

Now, it's not sufficient to just take the square root of all the values we've already computed – now that we have a new guess, we need to re-compute k. Before we dive in, let's make one quick simplification:

$$f = \sqrt{\frac{k}{M}} = \frac{\sqrt{k}}{\sqrt{M}}$$

And since $\sqrt{k}$ is a constant just like k, we can rename $\sqrt{k}$ as some other constant. To keep things consistent, we'll just call this new constant k.

So our new educated guess is that frequency is inversely proportional to the square root of mass per unit length:

$$f = \frac{k}{\sqrt{M}}\ ? \qquad (7)$$

Now that we have a new guess, let's test it!

Just as before, we'll compute k using one of our observations. And using our new value of k, we'll compute our predicted frequency for our observed cases:

$$f = k/\sqrt{m}$$
$$84.4 = k/\sqrt{6.1\times10^{-2}}$$
$$k = 84.4 \cdot \sqrt{6.1\times10^{-2}}$$
$$\boxed{k = 20.845}$$

Now, check out Figure 25! Our new predictions are a much better match to our observations! Our max error has dropped to 4.2%![6]

SETUP 1
length = 60 cm
tension = 1600g

MASS/LENGTH (g/cm)	OBSERVED FREQUENCY (Hz)	f = k/M PREDICTION	% ERROR	f = k/√m PREDICTION	% ERROR
6.10x10⁻²	84.4	84.4	–	84.4	–
3.96x10⁻²	105.2	5.148/3.96x10⁻² = 130.0	19.1	20.845/√3.96x10⁻² = 104.8	0.4
2.33x10⁻²	137.0	5.148/2.33x10⁻² = 221.0	38.0	20.845/√2.33x10⁻² = 136.6	0.3
8.86x10⁻³	222.0	5.148/8.86x10⁻³ = 581.1	61.8	20.845/√8.86x10⁻³ = 221.5	0.2
5.97x10⁻³	272.5	5.148/5.97x10⁻³ = 862.4	68.4	20.845/√5.97x10⁻³ = 269.8	1.0
4.54x10⁻³	324.0	5.148/4.54x10⁻³ = 1134.0	71.4	20.845/√4.54x10⁻³ = 309.4	4.7

Figure 25 | Testing educated guesses on our observed mass and frequency data. Our initial geuss - f = k/M doesn't do so well - especially as mass decreases. Our second guess f=k/√M does much better!

Now, to be really sure here, we need to make predictions for the string setup we haven't tested yet. Let's compute k using our single observation at this length and tension:

$$f = k/\sqrt{m}$$
$$123.6 = k/\sqrt{6.10\times10^{-2}}$$
$$k = 123.6 \cdot \sqrt{6.10\times10^{-2}}$$
$$\boxed{k = 30.53}$$

And using our new formula (7), we'll compute frequencies for our experimental setup - shown in Figure 26.

And now, making observations for these mass values, we see quite good agreement between our predictions and observations.

SETUP 2
length = 40 cm
tension = 1600g

MASS/LENGTH (g/cm)	OBSERVED FREQUENCY (Hz)	PREDICTED FREQUENCY (Hz)	% ERROR
6.10x10⁻²	123.6	123.6	–
3.96x10⁻²	155.7	30.53/√3.96x10⁻² = 153.4	1.5
2.33x10⁻²	208.5	30.53/√2.33x10⁻² = 200.0	4.3
8.86x10⁻³	328.4	30.53/√8.86x10⁻³ = 324.3	1.3
5.97x10⁻³	406.3	30.53/√5.97x10⁻³ = 395.1	2.8
4.54x10⁻³	470.0	30.53/√4.54x10⁻³ = 453.1	3.7

Figure 26 | Testing educated guesses on our unobserved mass and frequency data. Our second guess f=k/√M makes great predictions on our second string setup!

We have found Mersenne's third and final law!

Through careful experimentation and solving a few puzzles, we have discovered three very specific hidden connections between mathematics and the universe we live in.

Three scientific laws.

Now, after all that work, have we gained anything? Are our three discoveries just nice things to know about the world we live in, or is there more here? Can Mersenne's Laws give us any deeper insight into the mysteries we've seen so far?

Remember that we began our journey with a simple question from the ancient Greek mathematician Pythagoras. Why do some combinations of vibrating strings sound good together, and others don't?

5 Also, this is clearly legit becuase of *magic handwaving*.
6 woohoo!

And back in part 2, we discovered the incredible fact that there is, of all things, a mathematical answer to this question. Strings sound good together when the ratio of their lengths is simple and the ratio of the square roots of their tensions is simple.

Now, as remarkable as this discovery is – it doesn't fully answer the question posed by Pythagoras. These mathematical relationships do a great job telling us when two strings will sound good together, but don't really tell us *why* our two strings sound good together.

Digging deeper, we followed a hunch from the 16th century Italian mathematician Giambattista Benedetti, that the tone we hear from a vibrating string is direct result of how quickly it moves back and forth – it's frequency.

Focusing on frequency, we then discovered Mersenne's laws, allowing us to directly mathematically link the things that Pythagoras was paying attention to, a string's length and tension, to it's frequency.

Now, do Mersenne's Laws get us any closer to answering the question posed by Pythagoras - why some strings sound good together, and others don't?

Can Mersenne's Laws give us a better answer Pythagoras's question?[7]

If frequency really is the right thing to pay attention to, then maybe the frequencies of the strings that Pythagoras found to sound good together (shown in Tables 6 and 7) have something in common.

We can test this idea using Mersenne's Laws.

We'll start with one observed frequency at a length of 60 cm and a tension of 4000 g. Using our high speed camera setup, we measure a frequency of 129.6 Hz, as shown in Tables 6 and 7. Then, using Mersenne's Laws, we can compute the expected frequency of all the other lengths and tensions in our tables.

Alright, for one last time, I'm turning it over to you.

What frequencies do Mersenne's Laws predict for the string lengths and tensions in Tables 6 and 7?

Do the frequencies of strings that sound good together have anything in common?

Can Mersenne's Laws give us a better answer to the mystery of Pythagoras - why some vibrating strings sound good together, while others don't?

7 like 2000 years later...

Mersenne's Laws

$$f = \frac{k}{L} \tag{1}$$

Mathematical connection between length and frequency

$$f = k\sqrt{T} \tag{2}$$

Mathematical connection between tension and frequency

$$f = \frac{k}{\sqrt{M}} \tag{3}$$

Mathematical connection between mass and frequency

Figure 26 | Finally! We have found Mersenne's Laws. Three connections between mathematics and the universe we live in.

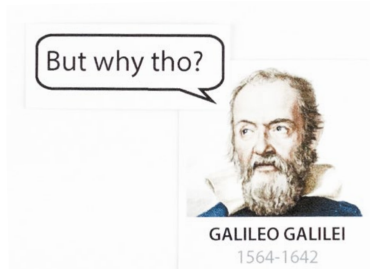

Figure 27 | Seriously, Why? We've discovered some really cool stuff so far, but have we really answered Pythagoras' original question - why some strings sound good together, while others don't?

STRING 2 LENGTH (cm)	FREQUENCY (Hz)	STRING 1 LENGTH (cm): 70	65	60	55	50	45	40	35	30	25	20
				129.6								
70		✓							✓			
65			✓									
60	129.6			✓			✓	✓		✓		✓
55					✓							
50						✓		✓		✓	✓	✓
45				✓			✓			✓		
40				✓		✓		✓		✓		✓
35		✓							✓			
30				✓		✓	✓	✓		✓		✓
25						✓					✓	✓
20				✓		✓		✓		✓	✓	✓

Table 6 | Frequencies and Sounds. Good sounding string combinations are shown with green checks. What frequencies to Mersenne's laws predict for each length? Is there a connection between the frequencies of good-sounding string combinations?

		STRING 1 TENSION (kg)																								
		1.6	1.8	2.0	2.2	2.4	2.6	2.8	3.0	3.2	3.4	3.6	3.8	4.0	4.2	4.4	4.6	4.8	5.0	5.2	5.4	5.6	5.8	6.0	6.2	6.4
	FREQUENCY (Hz)													129.6												
STRING 2 TENSION (kg)	1.6		✓										✓														✓
	1.8			✓							✓									✓							
	2.0				✓																						
	2.2					✓																					
	2.4						✓															✓					
	2.6							✓																			
	2.8								✓																		
	3.0									✓																	
	3.2			✓							✓									✓							
	3.4											✓															
	3.6		✓										✓														✓
	3.8													✓													
	4.0	129.6													✓												
	4.2															✓											
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	4.8																		✓								
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	6.2																									✓	
	6.4		✓										✓														✓

Table 6 | Frequencies and Sounds. Good sounding string combinations are shown with green checks. What frequencies to Mersenne's laws predict for each tension? Is there a connection between the frequencies of good-sounding string combinations?